Section One: Calculator-free (40 Marks)

Question 1 (4 marks)

Determine the domain and range of $f(g(x))$, given that $f(x)=\sqrt{x}$ and $g(x)=4-2^x$

Solution
$f(g(x))=f(4-2^x)$ $=\sqrt{4-2^x}$ Domain: We need $4-2^x \geq 0$, i.e. $2^x \leq 4$, i.e. $x \leq 2$. Range: $0 \leq y < 2$
✓ determines $f(g(x))$ correctly ✓ correctly identifies requirement that $4-2^x \geq 0$ ✓ correctly states domain ✓ correctly states range

Question 2 (4 marks)

Differentiate the following, without simplifying:

(a) $y=e^{2x-x^4}$ (2 marks)

Solution
Derivative: $(2-2x)e^{2x-x^2}$
Specific behaviours
✓ differentiates $(2x-2x^2)$ part ✓ e^{2x-x^2} remains in solution

(b) $y=\dfrac{5x}{x^2+4}$ (2 marks)

Solution
Derivative: $\dfrac{(x^2+4)5-(2x)(5x)}{(x^2+4)^2}$
Specific behaviours
✓ applies quotient rule correctly ✓ correctly differentiates each part

or

Solution
$y=5x(x^2+4)^{-1}$ so $y'=5(x^2+4)^{-1}+5x(-2x)(x^2+4)^{-2}$
Specific behaviours
✓ correctly applies chain rule

Question 3 (3 marks)

The probabilities of two events A and B are given by: $P(A)=0.6$ and $P(B)=0.3$
Calculate $P(A \cup B)$, given that A and B are independent.

Solution
$P(A \cup B)=P(A)+P(B)-P(A \cap B)$ and $P(A \cap B)=P(A)\times P(B)$ by independence So $P(A \cup B)=0.6+0.3-0.6\times 0.3=0.72$
Specific behaviours
✓ selects appropriate rule from formula sheet ✓ uses multiplication rule for independence ✓ substitutes and calculates probability

Question 4 (5 marks)

Find the maximum and minimum values over the interval $1 \leq x \leq 4$ of the function

$$f(x)=x+\frac{4}{x^2}$$

Solution
The function is continuous and differentiable in the interval $1 \leq x \leq 4$ and so the extreme values occur at the end points or at critical points. $f'(x)=1-\frac{8}{x^3}=0$ when $x=2$ and $f(2)=2+\frac{4}{2^2}=3$ Also $f(1)=1+\frac{4}{1^2}=5$ and $f(4)=4+\frac{4}{4^2}=4\frac{1}{4}$ So $f_{max}=5$ and $f_{min}=3$.
Specific behaviours
✓ correctly differentiates ✓ solves $f'(x)=0$ ✓ evaluates $f(2)$ ✓ evaluates $f(1)$ and $f(4)$ ✓ states maximum and minimum

Question 5 **(4 marks)**

Solve for x in the equation

$$\frac{3}{x}+\frac{4x}{1+2x}=2$$

Solution
$\dfrac{3(1+2x)+4x^2}{x(1+2x)}=2$ $3(1+2x)+4x^2=2x(1+2x)$ $3+6x+4x^2=2x+4x^2$ $3+4x=0$, $x=-\dfrac{3}{4}$
Specific behaviours
✓ recognises common denominator   ✓ multiplies by common denominator correctly   ✓ simplifies   ✓ states correct solution

Question 6 **(4 marks)**

Determine the following integrals:

(a) $\displaystyle\int \frac{x^2-1}{(x^3-3x)^2}\,dx$ (2 marks)

Solution
$=\dfrac{1}{3}\displaystyle\int\frac{3x^2-3}{\left(x^3-3x\right)^2}dx=\dfrac{1}{3}\int\left(x^3-3x\right)^{-2}(3x^2-3)\,dx$ $=\dfrac{1}{3}\dfrac{\left(x^2-3x\right)^{-1}}{(-1)}+C\ =-\dfrac{1}{3\left(x^3-3x\right)}+C$
Specific behaviours
✓ expresses integral in terms of $\int[f(x)]^n f'(x)\,dz$ ✓ integrates correctly and adds constant

(b) $\displaystyle\int_0^5 e^{-2x}\,dx$ (2 marks)

Solution
$\displaystyle\int_0^5 e^{-2x}dx=\left(-\frac{1}{2}e^{-2x}\right)\Big\vert_{x=0}^{x=5}$ $=\dfrac{1}{2}\left(-e^{-10}+e^{0}\right)=\dfrac{1}{2}\left(1-e^{-10}\right)$
Specific behaviours
✓ finds the integrand   ✓ substitutes limits of integration and simplifies

Question 7 **(5 marks)**

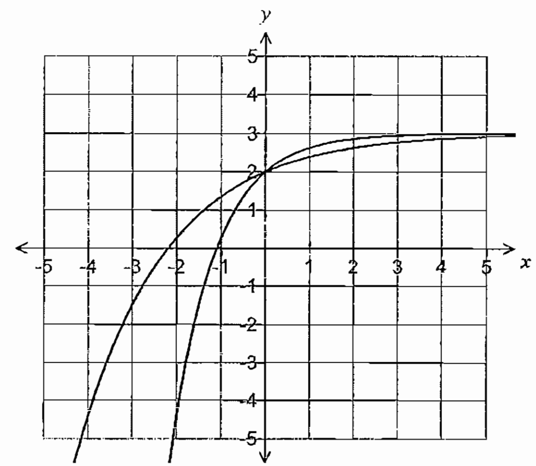
Solve the system of equations

$$\begin{aligned} x + 3y + z &= 2 \\ 2x + 5y + 3z &= 11 \\ 4x + 3y + 2z &= 16 \end{aligned}$$

Solution
For example: $-y+z=7$ Eq2 - 2Eq1 $\to$ Eq2 $-9y-2z=8$ Eq3 - 4Eq1 $\to$ Eq3 $11z=55$ 9Eq2 - Eq3 So $z=5$ Substitution gives $y=-2$ and $x=3$
Specific behaviours
✓✓ eliminates one variable from two pairs of equations ✓✓✓evaluates each of the variables correctly

Question 8 **(4 marks)**

The graph of $y = ae^{bx} + c$ is shown below. The graph passes through the point $(0, 2)$, and $y \to 3$ as $x \to \infty$



(a) Is b positive or negative? Justify your answer. (1 mark)

Solution
Since $y \to 3$ as $x \to \infty$, $e^{bx} \to 0$ as $x \to \infty$. So b must be negative.
Specific behaviours
✓ gives logical argument as to why b is negative

(b) Evaluate a and c. (2 marks)

Solution
Since $y \to 3$ as $x \to \infty$; $c = 3$. Since $y(0) = ae^0 + c = a + c = 2$, $a = -1$.
Specific behaviours
✓ evaluates c ✓ evaluates a

(c) Sketch on the same axes the graph of $y = ae^{2bx} + c$. (1 mark)

Solution
See graph above.
Specific behaviours
✓ draws graph with correct shape for $x > 0$ and $x < 0$, relative to the original graph

Question 9 **(7 marks)**

Determine all turning points and points of inflection of the function $f(x) = 2x^3 - 3x^2 - 12x + 20$, and use these to sketch its graph.

Solution

If $f(x) = 2x^3 - 3x^2 - 12x + 20$, then $f'(x) = 6x^2 - 6x - 12$ and $f''(x) = 12x - 6$

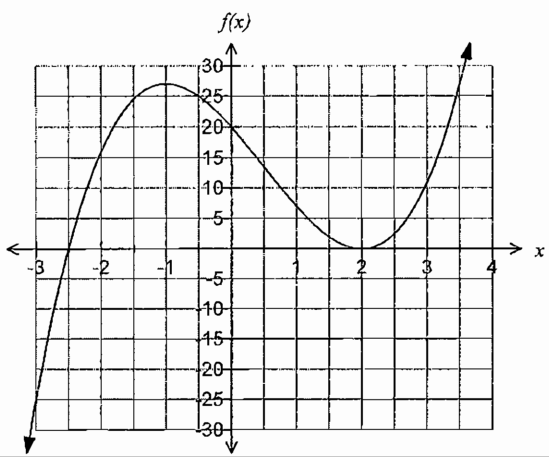
$6x^2 - 6x - 12 = 0 \Rightarrow 6(x-2)(x+1) = 0$

So the critical points occur at $x = 2$ and $x = -1$.

$12x - 6 = 0 \Rightarrow x = \frac{1}{2}$, where the point of inflection will be found.

Now $f(2) = 0$, $f(-1) = 27$ *and* $f(\frac{1}{2}) = \frac{27}{2}$, $f(0) = 20$

So the graph is



Specific behaviours

✓ determines $f'(x)$
✓ determines $f''(x)$
✓ finds critical points
✓ finds the point of inflection
✓ graph passes through the correct y-intercept
✓ graph passes through appropriate range of x values for intercept, i.e. $(-3$ to $-2)$
✓ correct shape of graph